The Wave Equation and the Vibrating String Model

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1 Discrete Derivation

Our goal is to be able to model the action of a vibrating string over time. The general model will be formed as shown in Figure (1).

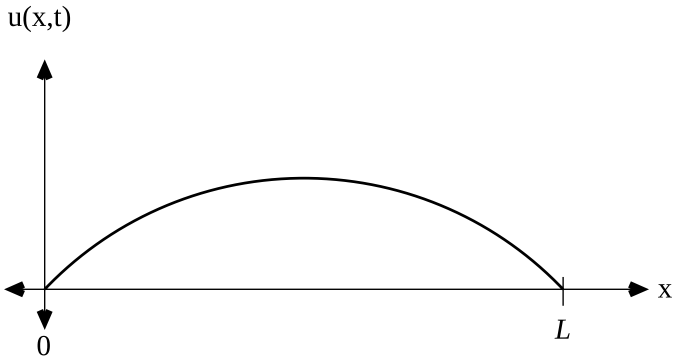

Figure 1: Model and naming convention.

We will invoke Newton's Second Law of Motion to the string. So first, we shall consider a tiny element of the string, as described in Figure (2). Here, we are looking at the interval $x \in (x, x + \Delta x)$. We define $\rho(x)$ to be the mass density at the point x on the string, $T(x,t)$ to be the tension in the string, and also $\theta(x,t)$ to be the angle between the string and a horizontal. Note then that

$$\tan\theta(x,t) = \frac{\partial u}{\partial x}. \tag{1.1}$$

We note that the mass of this small piece of string is about

$$\rho(x)\sqrt{\Delta x^2 + \Delta u^2}. \tag{1.2}$$

The forces that we have acting on this piece of the string are the tension to the right and the tension to the left, which have magnitudes $T(x + \Delta x, t)$ and

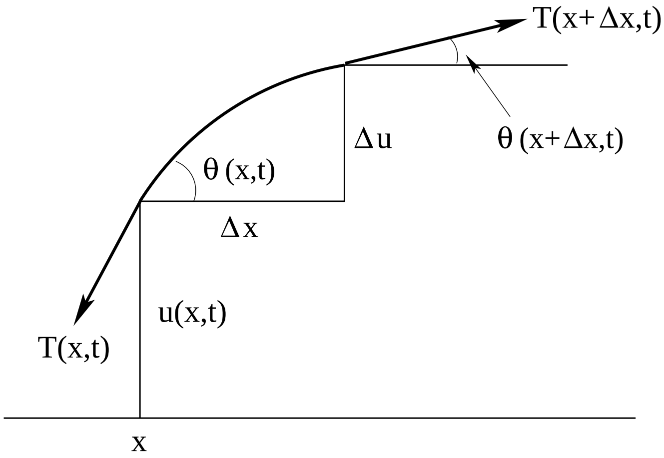

Figure 2: Examining a small piece of the string.

$T(x,t)$, respectively. Also, these forces act at angles $\theta(x+\Delta x,t)$ and $\theta(x,t)$, respectively. So we can write these into Newton's Second Law of Motion. We know that

$$a = \frac{\partial^2 u}{\partial t^2}, \tag{1.3}$$

$$m = \rho(x)\sqrt{\Delta x^2 + \Delta u^2} \tag{1.4}$$

The vertical forces are formed from the tension placed on the string, so we use $\sin\theta$ to find the vertical aspect of these forces, and combine them to reach

$$F = T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t), \tag{1.5}$$

since the left side of the string element wants to move downwards, and the right part of the string element wants to move upwards. So the vertical component of Newton's Law becomes

$$\begin{aligned}
\rho(x)\sqrt{\Delta x^2 + \Delta u^2}\frac{\partial^2 u}{\partial t^2} &= T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t), \\
\rho(x)\Delta x\sqrt{1+\frac{\Delta u^2}{\Delta x^2}}\frac{\partial^2 u}{\partial t^2} &= T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t), \\
\rho(x)\sqrt{1+\frac{\Delta u^2}{\Delta x^2}}\frac{\partial^2 u}{\partial t^2} &= \frac{T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t)}{\Delta x}.
\end{aligned} \tag{1.6}$$

Note that $\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x}$, so the left-hand-side of equation (1.6) becomes

$$\rho(x)\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2}\frac{\partial^2 u}{\partial t^2}. \tag{1.7}$$

Also, note that

$$\lim_{\Delta x \to 0} \frac{T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t)}{\Delta x} = \frac{\partial}{\partial x}\left(T(x,t)\sin\theta(x,t)\right) \tag{1.8}$$

Combining equations (1.7) and (1.8), we reach

$$\begin{aligned} \rho(x)\sqrt{1+(u_x)^2}\; u_{tt} &= \frac{\partial}{\partial x}\left(T(x,t)\sin\theta(x,t)\right) \\ &= T_x(x,t)\sin\theta(x,t) + T(x,t)\theta_x(x,t)\cos\theta(x,t) \end{aligned} \tag{1.9}$$

For this model, we consider only small vibrations in the string, as common in stringed musical instruments. By small vibrations, it is meant that the maximum displacement of the string is small, and so we realize that $\theta(x,t)$ will also be quite small. This implies that $\tan\theta$, and hence u_x, will also be quite small. From this, we can express that

$$\sqrt{1+u_x^2} \approx 1, \tag{1.10}$$

$$\sin\theta \approx \tan\theta = u_x, \tag{1.11}$$

$$\cos\theta \approx 1, \tag{1.12}$$

$$\theta_x = u_{xx}. \tag{1.13}$$

Using equations (1.10)–(1.13), we reduce equation (1.9) to

$$\rho(x)u_{tt} = T_x(x,t)u_x + T(x,t)u_{xx}. \tag{1.14}$$

Let us now consider the horizontal component of Newton's Second Law. We assume for our model that there are only transverse vibrations, and so the string does not move horizontally, but only vertically. So we know that the total horizontal force must be zero. We again note that the tension on the left of the string element wants to move leftwards, and vice versa, so we can express the horizontal forces as

$$T(x+\Delta x,t)\cos\theta(x+\Delta x,t) - T(x,t)\cos\theta(x,t) = 0 \tag{1.15}$$

As we divide by Δx and take the limit as $\Delta x \to 0$, equation (1.15) yields

$$\begin{aligned} \lim_{\Delta x \to 0} \frac{T(x+\Delta x,t)\cos\theta(x+\Delta x,t) - T(x,t)\cos\theta(x,t)}{\Delta x} &= \frac{\partial}{\partial x}\left(T(x,t)\cos\theta(x,t)\right) \\ &= 0 \end{aligned} \tag{1.16}$$

We again note here that since we are modeling only small vibrations, we find that $\cos\theta \approx 1$, and that $T_x \approx 0$, because there will not be much stretching occuring in the string, in a horizontal fashion. So what we find is that T will be a function only of time, as the overall tension will be governed by the tension put on the ends of the string, which is only a function of t. Since we know this, we can further reduce equation (1.14) to

$$\begin{aligned} \rho(x)u_{tt} &= T(t)u_{xx} \\ u_{tt} &= \frac{T}{\rho}u_{xx} \end{aligned} \tag{1.17}$$

For simplicity, we will consider the string to be homogeneous in density in this model, and also that T will be constant. It should be obvious that we define boundary conditions to be

$$u(0,t) = 0, \tag{1.18}$$
$$u(L,t) = 0. \tag{1.19}$$

We will also need two initial conditions, since we have a second partial in time. We simply define these to be

$$u(x,0) = f(x) \tag{1.20}$$
$$u_t(x,0) = g(x) \tag{1.21}$$

2 Lagrangian Derivation

As another method of derivation, we will form the Lagrangian for our system, and extremizing using the Euler-Lagrange equation, we will show the same outcome as the previous section. First, we note that the kinetic energy of a single point in the system is

$$m\frac{ds^2}{dt} = \frac{\rho(x)}{2}u_t^2. \tag{2.1}$$

So the total kinetic energy in the system is

$$\int_0^L \frac{\rho(x)}{2}u_t^2 \; dx. \tag{2.2}$$

We can suppose that on the string, the force generated will be proportional to the amount that the string is stretched, and the proportion we will call T. The amount the string is stretched by, with equilibrium length L, is given by

$$\int_0^L \sqrt{1+u_x^2} \; dx - L = \int_0^L \left(\sqrt{1+u_x^2} - 1\right) \; dx. \tag{2.3}$$

Then our potential energy is

$$\int_0^L T\left(\sqrt{1+u_x^2} - 1\right) \; dx. \tag{2.4}$$

Forming the "action integral," we get

$$\int_0^\infty \int_0^L \left(\frac{\rho(x)}{2}u_t^2 - T\left(\sqrt{1+u_x^2} - 1\right)\right) \; dx \; dt$$
$$= \int_0^\infty \int_0^L f(u, u_x, u_t, x, t) \; dx \; dt. \tag{2.5}$$

Now, we invoke the Euler-Lagrange equation for two variables,

$$\frac{\partial f}{\partial u} - \frac{d}{dt}\frac{\partial f}{\partial u_t} - \frac{d}{dx}\frac{\partial f}{\partial u_x} = 0. \tag{2.6}$$

From equation (2.5), we see that

$$\frac{\partial f}{\partial u} = 0, \tag{2.7}$$

$$\frac{d}{dt}\frac{\partial f}{\partial u_t} = \frac{d}{dt}\left(\rho(x)u_t\right) = \rho(x)u_{tt}. \tag{2.8}$$

Recall from section (1) that since the amplitude of the vibrations of the string are considered very small, then $\sqrt{1+u_x^2} \approx 1$. Then we also get

$$\begin{aligned}\frac{d}{dx}\frac{\partial f}{\partial u_x} &= \frac{d}{dx}\frac{\partial}{\partial u_x}\left(-T\left(sqrt1 + u_x^2 - 1\right)\right)\\ &= -T\frac{d}{dx}\left(\frac{u_x}{\sqrt{1+u_x^2}}\right)\\ &\approx -T\frac{d}{dx}u_x = -Tu_{xx}\end{aligned} \tag{2.9}$$

Substituting equations (2.7)–(2.9) back into equation (2.6), we reach

$$u_{tt} = \frac{T}{\rho}u_{xx}. \tag{2.10}$$