The Clock Problem

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1 The Problem

The clock is quickly approaching 7:05. Just afterwards, the hands of the clock will form a perfectly straight line. At what *exact* time does this occur?

2 The Model

To find the solution to this, we need expressions to represent the positions of the hands of the clock at a given time. Then, we can relate these expressions in some way to determine the time at which the hands are forming a straight line.

We can express the angles in the clock using either radians or degrees. Here, I will choose to use radians, as I tend to work more naturally with them. As you will find out, the choice is arbitrary.

We can expect the solution to lie somewhere between 7:05 and 7:10. First, let x be the time, in seconds, since 7:00 on the nose. The position of the minute hand will be exactly at 12:00, and start moving clockwise (duh!). For each 60 seconds, the minute hand moves $\frac{1}{60}$th of the way around the clock. Using radians, the angle of traveling around the entire clock is 2π, which is the same as $360°$, and so half way around the clock is π radians.

Note then that it takes 3600 seconds for the minute hand to sweep the entire clock face. So the angle of position of the minute hand at time x seconds is given by

$$2\pi\frac{x}{3600}. \tag{1}$$

Now we need to try the same thing for the hour hand. The hour hand moves slower than the minute hand, of course, and so we need to model that. In one hour, or 360 seconds, the hour hand moves $\frac{1}{12}$ of the way around the clock. Also recall that we said $x = 0$ represented 7:00, so the hour hand is right over the seven, or $\frac{7}{12}$ths past midnight. Then the position of the hour hand is

$$2\pi\frac{x}{3600(12)} + 2\pi\frac{7}{12}. \tag{2}$$

3 The Solution

Okay, so now we can model where the hands are at a given time after 7:00. But we want them to be in a perfectly straight line. Remember above how π radians was half of a circle? That's how we want the hands to be! Also, the hour hand will be past the minute hand. So it looks like we want

$$2\pi\frac{x}{3600} + \pi = 2\pi\frac{x}{3600(12)} + 2\pi\frac{7}{12}. \tag{3}$$

Perfect! Now we have a single equation with a single unknown. This ought to be easy! First, I see that everything has a factor of π, so we can divide both sides by that. That's good, we don't really like irrationals, right? Also, I am going to divide everything by 2. Then this gives us

$$\frac{x}{3600} + \frac{1}{2} = \frac{x}{3600(12)} + \frac{7}{12}. \tag{4}$$

I don't really like the fractions, either, so I will multiply everything by 3600(12) to remove them. This gives us

$$12x + 3600(6) = x + 3600(7). \tag{5}$$

Looking better! Let's subtract x from both sides, and also subtract 3600(6) from both sides. Now we get

$$11x = 3600(7) - 3600(6) = 3600. \tag{6}$$

So we finally divide both sides by 11 and get

$$x = \frac{3600}{11} = 327\frac{3}{11}. \tag{7}$$

Now, remember that x is in seconds, so we want to pull out the minutes. Sixty seconds in a minute, so this gives

$$x = 5\text{min}.27\frac{3}{11}\text{sec}. \tag{8}$$

So the clock hands form a perfectly straight line at 7:05:25$\frac{3}{11}$. Easy, right?